SHARP BOUNDS IN THE BINARY ROY MODEL

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Abstract. We derive the empirical content of an instrumental variables model of sectorial choice with discrete outcomes. Assumptions on selection include the simple, extended and generalized Roy models. The derived bounds are nonparametric intersection bounds and are simple enough to lend themselves to existing inference methods. Identification implications of exclusion restrictions are also derived.

Keywords: treatment effect, discrete outcomes, sectorial choice, partial identification, intersection bounds.

JEL subject classification: C21, C25, C26

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Introduction

A large literature has developed since Heckman and Honoré (1990) on the empirical content of the Roy model of sectorial choice with sector specific unobserved heterogeneity. Most of this literature, however, concerns the case of continuous outcomes and many applications, where outcomes are discrete, fall outside its scope. They include analysis of the effects of different training programs on the probability of renewed employment, of competing medical treatments or surgical procedures on the probability of survival, of higher education on the probability of migration and of competing policies on schooling decisions in developing countries among numerous others. The Roy model is still highly relevant to those applications, but very little is known of its empirical content in such cases. Sharp bounds are derived in binary outcome models with a binary endogenous regressor in Chesher (2010), Shaikh and Vytlacil (2011), Chiburis (2010), Jun, Pinkse, and Xu (2010) and Mourifié (2011) under a variety of assumptions, which all rule out sector specific unobserved heterogeneity. Finally, Heckman and Vytlacil (1999) derive identification conditions in a parametric version of the binary Roy model.

We consider three distinct versions of the binary Roy model: the original model, where selection is based solely on the probability of success; the extended Roy model (in the terminology of Heckman and Vytlacil (1999)), where selection depends on the probability of success and a function of observable variables (sometimes called "nonpecuniary component"); and the generalized Roy model (in the terminology of Heckman and Honoré (1990)), with selection specific unobservable heterogeneity. When considering the generalized Roy model, we further distinguish restrictions on the selection equation and restrictions on the joint distribution of sector specific unobserved heterogeneity. We specifically consider the case, where selection variables are independent of sector specific unobserved heterogeneity and the case, where sector specific unobserved heterogeneity follows a factor structure proposed in Aakvik, Heckman, and Vytlacil (2005).

Following Heckman, Smith, and Clements (1997), we apply results from optimal transportation theory to derive sharp bounds on the structural parameters, from which a range of treatment parameters can be derived. More specifically, we apply Theorem 1 of Galichon and Henry (2011) (equivalently Theorem 3.2 of Beresteanu, Molchanov, and Molinari (2011)) to derive bounds for the generalized discrete Roy model. The latter Theorem was recently applied in a similar context by Chesher, Rosen, and Smolinski (2011) to derive sharp bounds for instrumental variable models of discrete choice. We spell out the point identification implications of the bounds under certain exclusion restrictions. The bounds are simple enough to lend themselves to existing inferential methods, specifically Chernozhukov, Lee, and Rosen (2009) and Andrews and Shi (2011) in the instrumental variables case.

The remainder of the note is organized as follows. Section 1 clarifies the analytical framework. In Section 2, sharp bounds are derived for the binary Roy model, when selection depends only on the probability of success and possibly on observable variables. Identification implications are spelled out under exclusion restrictions. Section 3 considers the generalized binary Roy model and the last section concludes.

1. Analytical framework

We adopt the framework of the potential outcomes model $Y = Y_1 D + Y_0(1-D)$, where Y is an observed outcome, D is an observed selection indicator and Y_1, Y_0 are unobserved potential outcomes. Heckman and Vytlacil (1999) trace the genealogy of this model and we refer to them for terminology and attribution. Potential outcomes are as follows:

$$Y_d = 1\{Y_d^* > 0\} = 1\{F(d, X_d, u_d) > 0\},\ d = 1, 0, \tag{1.1}$$

where $1\{.\}$ denotes the indicator function and F is an unknown function of the vector of observable random variables X_d and unobserved random variable u_d. We make the following assumptions throughout Sections 2 and 3.

Assumption 1 (Weak separability)**.** *The functions $F(d, X_d, u_d)$, d=1,0, both have weakly separable errors. As shown in Vytlacil (2002), potential outcomes can then be written $Y_d = 1\{f_d(X_d) > u_d\}$ without loss of generality.*

Assumption 2 (Regularity)**.** *The sector specific unobserved variables u_d, $d = 1, 0$, are uniformly continuous with respect to Lebesgue measure, so that they may be assumed without loss of generality to be distributed uniformly on $[0, 1]$.*

The normalization of Assumption 2 is very convenient, since it implies $f_d(x_d) = \mathbb{E}(Y_d|x_d, z)$ and bounds on treatment effects parameters can be derived from bounds on the structural parameters f_1 and f_0.

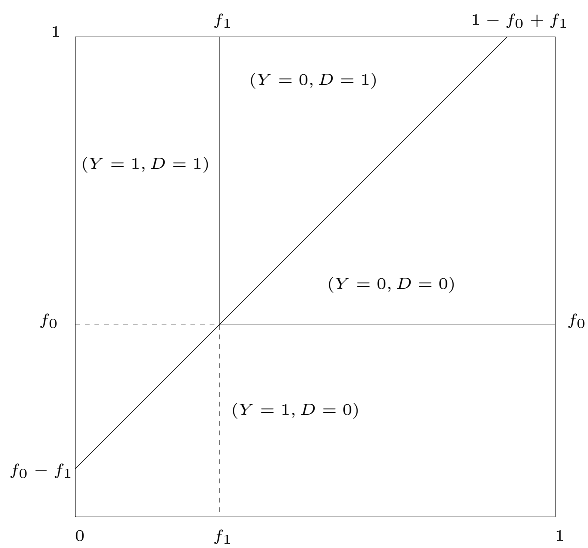
Assumption 3 (Instruments)**.** *Observable variables X_d, $d = 1, 0$, and instruments Z are independent of (u_1, u_0). Common components of X_1 and X_0 will be dropped from the notation in the remainder of the note and by slight abuse of notation, X_d will refer only to the variables that are excluded from the equation for Y_{1-d} and Z to variables that are excluded from both outcome equations (when the case arises).*

In all the note, we shall use the notation $\mathbb{P}(i, j|X)$ for $\mathbb{P}(Y = i, D = j|X)$ and $W = (Z, X_1, X_0)$, $\omega = (z, x_1, x_0)$.

2. Sharp bounds for the binary Roy and extended Roy models

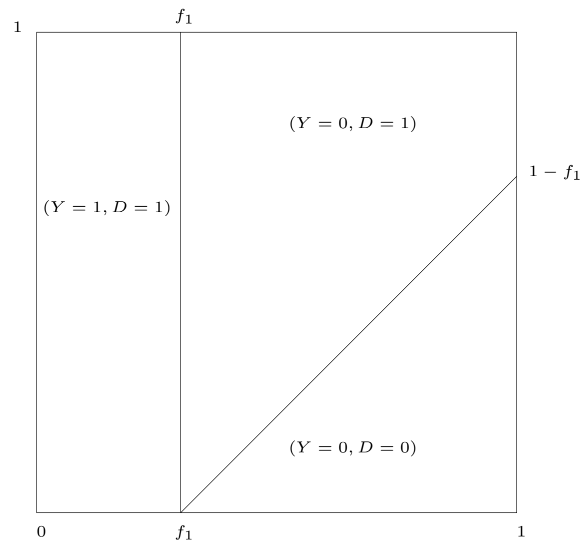
2.1. **Simple binary Roy model.** In the original model proposed by Roy (1951), the sector yielding the highest outcome is selected, i.e., $D = 1\{Y_1^* > Y_0^*\}$. In the binary case, this is equivalent to selecting the sector with the highest probability of success. The empirical content of the model under this selection rule is characterized in Figures 1 and 2.

FIGURE 1. Characterization of the empirical content of the simple binary Roy model in the unit square of the (u_1, u_0) space.



2.1.1. *Bounds on the structural functions.* For each value of the exogenous observable variables and each value of the pair (u_1, u_0), the outcome is uniquely determined. If the joint distribution were known, the likelihood of each of the potential outcomes $(Y = 1, D = 1)$, $(Y = 1, D = 0)$, $(Y = 0, D = 1)$ and $(Y = 0, D = 0)$ would be determined. However, only the marginal distributions of u_1 and u_0 are fixed, not the copula, so that only the probability of vertical and horizontal bands in Figures 1 and 2 are uniquely determined. Thus we see for instance that $f_1 = \mathbb{P}(Y = 1, D = 1)$ is identified when $f_0 = 0$ (as in Figure 2) and $f_0 = \mathbb{P}(Y = 1, D = 0)$ is identified when $f_1 = 0$ in a way that is akin to identification at infinity, as in Heckman (1990), when $f_i(x)$ follows a single index restriction. But in other cases (as in Figure 1), we only know $\mathbb{P}(Y = 1, D = 1) \leq f_1 \leq \mathbb{P}(Y = 1)$

Figure 2. Characterization of the empirical content of the simple binary Roy model in the unit square of the (u_1, u_0) space in case $f_0 = 0$.



and $\mathbb{P}(Y = 1, D = 0) \leq f_0 \leq \mathbb{P}(Y = 1)$. The following proposition, proved in the Appendix, shows that these bounds are jointly sharp.

Proposition 1 (Roy model)**.** *Under Assumptions 1-3, the following inequalities characterize the identified set for* (f_1, f_0) *under model (1.1) with* $D = 1\{Y_1^* > Y_0^*\}$*.*

$$\sup_{x_0} \mathbb{P}(1,1|x_1,x_0) \leq \quad f_1(x_1) \quad \leq \inf_{x_0} \Big[\mathbb{P}(1,1|x_1,x_0) + \mathbb{P}(1,0|x_1,x_0) 1\{f_0(x_0) > 0\} \Big] \tag{2.1}$$

$$\sup_{x_1} \mathbb{P}(1,0|x_1,x_0) \leq \quad f_0(x_0) \quad \leq \inf_{x_1} \Big[\mathbb{P}(1,0|x_1,x_0) + \mathbb{P}(1,1|x_1,x_0) 1\{f_1(x_1) > 0\} \Big] \tag{2.2}$$

where the infima and suprema are taken over the domains of the excluded variables X_1 *or* X_0 *as indicated and when they exist.*

The validity of the bounds was shown above. To prove sharpness, we show in Appendix A that we can construct joint distributions for (u_1, u_0) such that each of the extreme points of the identified region for $(f_1(x_1), f_0(x_0))$ defined by (2.1) and (2.2) are attained. Since the bounds in Proposition 1 are obtained as intersections over the domains of the excluded variables, they are called "intersection bounds". They are also semiparametric in the non excluded variables. Inference on such bounds can be conducted with existing methods described in Chernozhukov, Lee, and Rosen (2009) or in Andrews and Shi (2011).

A simple implication of selection equation $D = 1\{Y_1^* > Y_0^*\}$ is that actual success is more likely than counterfactual success.

Assumption 4 (Roy model)**.** $\mathbb{E}(Y_d|D = d, X_1, X_0) \geq \mathbb{E}(Y_{1-d}|D = d, X_1, X_0)$ *for* $d = 1, 0$.

Under Assumption 4, omitting conditioning variables for ease of notation,

$$\begin{aligned} f_d &= \mathbb{E}[Y_d] \\ &= \mathbb{E}[Y_d|D = d]\mathbb{P}(D = d) + \mathbb{E}[Y_d|D = 1-d]\mathbb{P}(D = 1-d) \\ &\leq \mathbb{P}[Y = 1, D = d] + \mathbb{E}[Y_{1-d}|D = 1-d]\mathbb{P}(D = 1-d) \\ &= \mathbb{P}(Y = 1, D = d) + \mathbb{P}(Y = 1, D = 1-d). \end{aligned}$$

Moreover, if $f_d > 0$ and $f_{1-d} = 0$, $\mathbb{P}(D = 1-d) = 0$. This implies that

$$\mathbb{P}(1, d|x_1, x_0) \leq \mathbb{E}[Y_d|x_1, x_0] \leq \mathbb{P}(1, d|x_1, x_0) + \mathbb{P}(1, 1-d|x_1, x_0)1\{\mathbb{E}[Y_{1-d}|x_1, x_0] > 0\}$$

characterizes the empirical content of the potential outcomes model $Y = Y_1 D + Y_0(1-D)$ in all generality (i.e., without weak separability and without assumptions on the dimension of unobservable heterogeneity). It also shows that the simple binary Roy model has no empirical content relative to (f_1, f_0) beyond Assumption 4. More precisely, the identified set for (f_1, f_0) under Assumptions 1-4 is the same as under Assumption 1-3 with Roy selection $D = 1\{Y_1^* > Y_0^*\}$. Indeed, bounds (2.1) and (2.1) still hold under Assumptions 1-4. They are also sharp, since $D = 1\{Y_1^* > Y_0^*\}$ implies Assumption 4.

Corollary 1. *The identified set for (f_1, f_0) under Assumptions 1-4 is characterized by inequalities (2.1) and (2.2).*

In case of exclusion restrictions, an immediate corollary to Proposition 1 gives conditions for identification of the outcome equations. This identification result is related to Heckman (1990)'s identification at infinity in the following sense: in the special case of a single index model, where $f_0(x_0) = \phi(x_0\beta)$, where ϕ is a distribution function and β is a conformable vector of parameters, if $x_{0j} \to -\infty$ for some element x_{0j} of x_0 such that $\beta_j > 0$, then $f_0(x_0) \to 0$ as required.

Corollary 2 (Identification)**.** *Under Assumptions 1-4, the following hold (writing $\omega = (z, x_1, x_0)$ as before).*

a. *If there is $x_0 \in Dom(X_0)$ such that $f_0(x_0) = 0$, then f_1 is identified over $Dom(X_1)$.*

b. *If there is $x_1 \in Dom(X_1)$ such that $f_1(x_1) = 0$, then f_0 is identified over $Dom(X_0)$.*

a'. *Take $x_1 \in Dom(X_1)$. If there is $x_0 \in Dom(X_0)$ such that $\mathbb{P}(1, 0|x_1, x_0) = 0$, then $f_1(x_1)$ is identified.*

b'. *Take $x_0 \in Dom(X_0)$. If there is $x_1 \in Dom(X_1)$ such that $\mathbb{P}(1, 1|x_1, x_0) = 0$, then $f_0(x_0)$ is identified.*

The existence of valid instruments or exclusion restrictions is often problematic in applications of discrete choice models. However, in the Roy model of sectorial choice with sector specific unobserved heterogeneity, it is natural to expect some sector specific observed heterogeneity as well. Such sector specific observed heterogeneity would provide exclusion restrictions in the form of variables affecting outcome equation for Y_d without affecting outcome equation for Y_{1-d}. Such exclusion restrictions would yield intersection bounds in Proposition 1. Of course, even if it exists, sector specific observed heterogeneity may not satisfy a. or b. of Corollary 2. However, the availability of an exclusion restriction as in a. or b. of Corollary 2 is consistent with the spirit of a model of sector specific heterogeneity.

2.1.2. *Bounds on the joint distribution of sector specific heterogeneity.* The bounds proposed in Proposition 1 are joint sharp bounds on the structural functions, hence on treatment effects. To derive them, we treated the joint distribution of sector specific heterogeneity as a nuisance parameter. One may also be interested in the empirical content of the model relative to sector specific heterogeneity. Since the distributions of u_1 and u_0 are both normalized and assumed uniform on $[0,1]$, the joint distribution satisfies Fréchet bounds:

$$\max(f_1(x_1)+f_0(x_0)-1,0) \leq \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1,x_0) \leq \min(f_1(x_1), f_0(x_0)).$$

The relevant question, therefore, is whether the Roy model assumption on selection $D = 1\{Y_1^* > Y_0^*\}$ holds empirical content relative to the distribution of unobserved sector specific heterogeneity beyond Fréchet bounds. On Figure 1, $\mathbb{P}(Y=1)$ is equal to the L-shaped region on the left side of the graph. The area of the left vertical band is f_1 and the area of the lower horizontal band is f_0. These two bands overlap on the lower left rectangle, whose area is equal to $\mathbb{P}(u_1 \leq f_1, u_0 \leq f_0)$. Hence $f_1 + f_0 = \mathbb{P}(Y=1) + \mathbb{P}(u_1 \leq f_1, u_0 \leq f_0)$. Adding conditioning variables, we have the following bounds on the joint distribution of sector specific heterogeneity:

$$\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1,x_0) = f_1(x_1) + f_0(x_0) - \mathbb{P}(Y=1|x_1,x_0). \tag{2.3}$$

This yields a sharper lower bound than the Fréchet bounds whenever $\mathbb{P}(Y=1|x_1,x_0) < 1$. Note however, that the above constraint no longer holds when we replace the Roy selection hypothesis $D = \{Y_1^* > Y_0^*\}$ by Assumption 4. Hence the conclusion that the latter two assumptions hold the same empirical content is valid when considering empirical content relative to the structural functions and the treatment effects, but not when considering empirical content relative to the distribution of unobserved heterogeneity.

2.2. **Extended binary Roy model.**

2.2.1. *Extended selection assumption.* Assumption 4 is very restrictive and recent research by Haultfoeuille and Maurel (2011) and Bayer, Khan, and Timmins (2011) on the Roy model with continuous outcomes has focused on an extended version according to the terminology of Heckman and Vytlacil (1999), where selection depends on $Y_1^* - Y_0^*$ and a function of observable variables $g(Z, X_1, X_0)$ sometimes called "non pecuniary component". We now investigate the implications of this selection assumption in the binary case.

Assumption 5 (Observable heterogeneity in selection)**.** $D = 1\{Y_1^* - Y_0^* > g(Z, X_1, X_0)\}$ *for some unknown function* g *of the vector of the observable variables* Z, X_1 *and* X_0.

Assumption 5 includes weak separability of the structural selection function in $Y_1^* - Y_0^*$ and $g(Z, X_1, X_0)$. The more general case without weak separability of the selection function is considered in Section 2.2.5. Under Assumptions 1-3 and 5, we may still characterize the empirical content of the model graphically, in Figures 3 and 4. We drop Z, X_1 and X_0 from the notation in the discussion below. For each value of (u_1, u_0), the outcome is uniquely determined by f_1, f_0 and g. Again, the missing piece to compute the likelihood of outcomes $\mathbb{P}(i, j)$, $i, j = 1, 0$, is the copula for (u_1, u_0). From the knowledge of the probabilities of horizontal and vertical bands in the (u_1, u_0) space, we can derive the sharp bounds on structural parameters f_1, f_0 and g. Four cases are considered below to explain the bounds, which are derived formally in Proposition 2.

a. Case where $g \geq f_1$ on Figure 4. The probability of outcome $(Y = 1, D = 0)$ is seen to be exactly equal to the area of the lower horizontal band. Hence $f_0 = \mathbb{P}(1, 0)$ is identified. Moreover, the area of the horizontal band $(f_0, f_0 - f_1 + g)$ is smaller than the probability of outcome $(Y = 1, D = 1)$. Hence $g \leq f_1 + \mathbb{P}(1, 1)$. Similar reasoning yields $\mathbb{P}(1, 1) \leq f_1 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 0)$.

FIGURE 3. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $0 \leq g < f_1$.

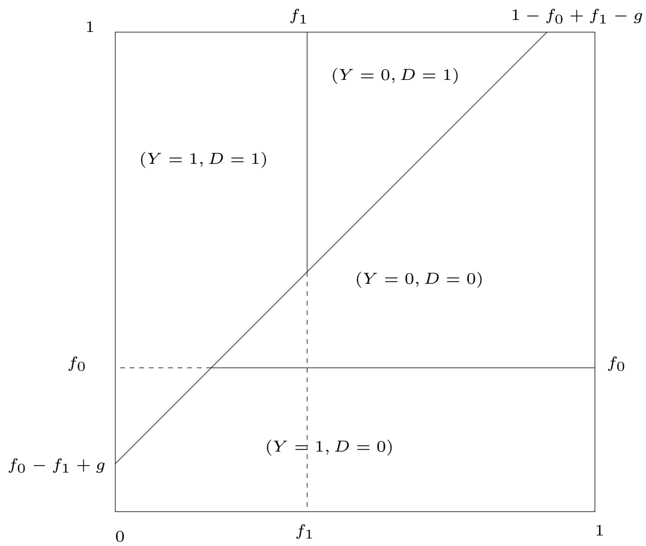
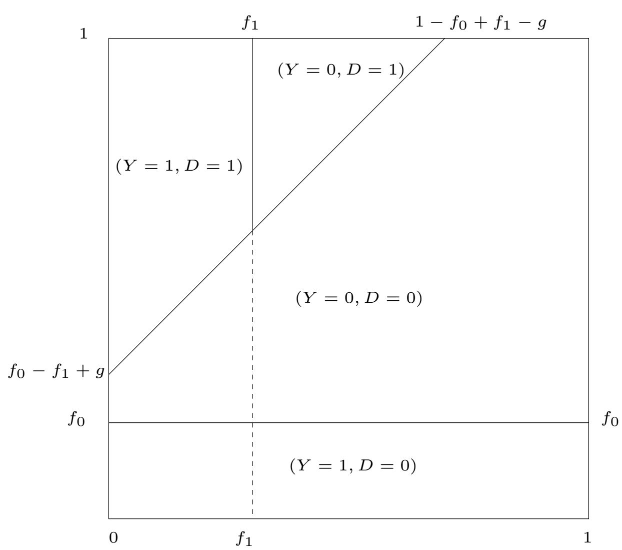


FIGURE 4. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $g \geq f_1$.



b. Case where $0 \leq g < f_1$ on Figure 3. The area of the lower horizontal band $(0, f_0 - f_1 + g)$ is smaller than the probability of outcome $(Y = 1, D = 0)$. Hence $g \leq f_1 - f_0 + \mathbb{P}(1, 0)$. Moreover, the area of the horizontal band $(0, f_0)$ is larger than the probability of outcome $(Y = 1, D = 0)$ and smaller than the probability of outcome $(Y = 1)$. Hence $\mathbb{P}(1, 0) \leq f_0 \leq \mathbb{P}(Y = 1)$. Finally, $\mathbb{P}(1, 1) \leq f_1 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 0)$ still holds.

c. Case where $-f_0 < g \leq 0$. Similarly to Case b., we obtain bounds $g \geq f_1 - f_0 + \mathbb{P}(1, 1)$, $\mathbb{P}(1, 0) \leq f_0 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 1)$ and $\mathbb{P}(1, 1) \leq f_1 \leq \mathbb{P}(Y = 1)$.

d. Case where $g \leq -f_0$. Similarly to Case a., $f_1 = \mathbb{P}(1, 1)$ is identified and $\mathbb{P}(1, 0) \leq f_0 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 1)$ and $g \geq -f_0 - \mathbb{P}(0, 1)$.

In addition, in both cases a. and b., where $g > f_1 - f_0$, corresponding to Figures 4 and 3, the marginal constraint on u_1 fixes the probability mass in the thin right vertical band to $f_0 - f_1 + g$. Hence the maximum probability mass that can be shifted to the left of f_1 is $p_{11} + p_{10} + p_{00} - (f_0 - f_1 + g)$, so that we have the additional constraint $f_0 \leq p_{11} + p_{10} + p_{00} - g$. Symmetrically, in case $g < f_1 - f_0$, we have the constraint $f_1 \leq g + p_{11} + p_{10} + p_{00}$. Since $g > f_1 - f_0$ also implies $f_1 \leq g + p_{11} + p_{10} + p_{00}$ and $g < f_1 - f_0$ also implies $f_0 \leq p_{11} + p_{10} + p_{00} - g$, the two constraints $f_0 \leq p_{11} + p_{00} + p_{10} - g$ and $f_1 \leq g + p_{11} + p_{10} + p_{01}$ always hold. Proposition 2 shows validity of the bounds discussed above for the triplet $(f_1(x_1), f_0(x_0), g(\omega))$.

Proposition 2 (Bounds for the extended binary Roy model)**.** *Under Assumptions 1-3 and 5, the following bounds for* (f_1, f_0, g) *hold (writing* $\omega = (z, x_1, x_0)$ *as before).*

$$\begin{aligned} f_1(x_1) \quad &\in \quad \Big[\sup_{z,x_0} \mathbb{P}(1,1|\omega), \inf_{z,x_0} [\mathbb{P}(1,1|\omega) + \mathbb{P}(0,0|\omega)1\{g(\omega) > 0\} \\ &\qquad + \min[\min(\mathbb{P}(1,0|\omega), f_0(x_0) + g(\omega))1\{g(\omega) > -f_0(x_0)\}), \\ &\qquad\qquad g(\omega) + \mathbb{P}(1,0|\omega) + \mathbb{P}(0,1|\omega)]]\Big], \\ f_0(x_0) \quad &\in \quad \Big[\sup_{z,x_1} \mathbb{P}(1,0|\omega), \inf_{z,x_1} [\mathbb{P}(1,0|\omega) + \mathbb{P}(0,1|\omega)1\{g(\omega) < 0\} \\ &\qquad + \min[\min(\mathbb{P}(1,1|\omega), f_1(x_1) - g(\omega))1\{g(\omega) < f_1(x_1)\}), \\ &\qquad\qquad \mathbb{P}(1,1|\omega) + \mathbb{P}(0,0|\omega) - g(\omega)]]\Big] \end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
g(\omega) \in \Big(\Big[-f_0(x_0) - \mathbb{P}(0,1|\omega), -f_0(x_0) \Big] \cup \Big[f_1(x_1) - f_0(x_0) - \mathbb{P}(1,1|\omega), \\
f_1(x_1) - f_0(x_0) + \mathbb{P}(1,0|\omega) \Big] \cup \Big[f_1(x_1), f_1(x_1) + \mathbb{P}(0,0|\omega) \Big] \Big) \\
\cap \Big[f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), \mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) + \mathbb{P}(0,0|\omega) - f_0(x_0) \Big]
\end{aligned} \tag{2.5}
$$

where the infima and suprema are taken over the domain of Z, X_1 or X_0 as indicated and when they arise.

2.2.2. *Identification implications of exclusion restrictions.* Simple identification conditions can be derived for f_1 and f_0 from the bounds of Proposition 2 under exclusion restrictions. However, it can be seen immediately that exclusion restrictions cannot identify $g(\,)$, since it would require $\mathbb{P}(Y=1,D=1|\omega)$, $\mathbb{P}(Y=0,D=1|\omega)$, $\mathbb{P}(Y=1,D=0|\omega)$ and $\mathbb{P}(Y=0,D=0|\omega)$ to simultaneously equal zero.

Corollary 3 (Identification)**.** *Under Assumptions 1-3 and 5, the following hold (writing $\omega = (z, x_1, x_0)$ as before).*

a. *If there is $z \in Dom(Z)$ and $x_0 \in Dom(X_0)$ such that $g(\omega) \leq -f_0(x_0)$, then $f_1(x_1) = \mathbb{P}(1,1|\omega)$ is identified.*

b. *If there is $z \in Dom(Z)$ and $x_1 \in Dom(X_1)$ such that $g(\omega) \geq f_1(x_1)$, then $f_0(x_0) = \mathbb{P}(1,0|\omega)$ is identified.*

a'. *Take $x_1 \in Dom(X_1)$. If there is $x_0 \in Dom(X_0)$ or $z \in Dom(Z)$ such that $\mathbb{P}(1,0|\omega) = \mathbb{P}(0,0|\omega) = 0$, then $f_1(x_1)$ is identified.*

b'. *Take $x_0 \in Dom(X_0)$. If there is $x_1 \in Dom(X_1)$ or $z \in Dom(Z)$ such that $\mathbb{P}(1,1|\omega) = \mathbb{P}(0,1|\omega) = 0$, then $f_0(x_0)$ is identified.*

2.2.3. *Sharp bounds in the extended Roy model.* When the object of interest is treatment parameters only, the three dimensional identification region defined by the sharp bounds on (f_1, f_0, g) is projected on the two-dimensional space (f_1, f_0) as follows.

Proposition 3 (Sharp bounds for the extended Roy model)**.** *Under Assumptions 1-3 and 5, the identified set for (f_1, f_0) is characterized by the following bounds, where λ takes the values 1 or 0 and $\varepsilon > 0$ is arbitrarily small:*

$$\begin{array}{rcccl} \mathbb{P}(1,1|\omega) & \leq & f_1(x_1) & \leq & \mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) + \lambda \max(0, \mathbb{P}(0,0|\omega) - \varepsilon) \\ \mathbb{P}(1,0|\omega) & \leq & f_0(x_0) & \leq & \mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega) + (1-\lambda) \max(0, \mathbb{P}(0,1|\omega) - \varepsilon) \end{array} \tag{2.6}$$

As in the case of the simple Roy model, the sharp bounds of Proposition 3 take the form of intersection bounds and inference can be conducted with existing methods.

If the object of interest is the non pecuniary component g, the three dimensional identification region is projected on the one-dimensional space for g into the single interval $[-\mathbb{P}(1,1|\omega) - \mathbb{P}(1,0|\omega), \mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega)]$, since the bounds in (2.4) cross at those values. In the presence of instruments (or exclusion restrictions), the projections on (f_1, f_0) and on g can be much tighter and the projection on (f_1, f_0) may even be reduced to a point, as in Corollary 3.

2.2.4. *Testing the Roy selection assumption.* As we have just seen, in the absence of exclusion restrictions, the identified region always contains the hyperplane $g = 0$, so that it is impossible to test the classical Roy selection hypothesis. However, in the presence of exclusion restrictions, the hypothesis $g(\omega) = 0$ may become testable. There is a non zero non pecuniary component in the selection equation if the hyperplane $g(\omega) = 0$ does not intersect the three dimensional identification region for $(f_1(x_1), f_0(x_0), g(\omega))$ defined by the bounds in Proposition 2. This implies the crossing of the intersection bounds in Proposition 1, in the sense that

$$\sup_{x_0,z} \mathbb{P}(1,1|x_1,x_0,z) \quad > \quad \inf_{x_0,z} \Big[\mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) 1\{f_0(x_0) > 0\}\Big]$$

or

$$\sup_{x_1,z} \mathbb{P}(1,0|\omega) \quad > \quad \inf_{x_1,z} \Big[\mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega) 1\{f_1(x_1) > 0\} \Big]$$

so that by Proposition 1, the simple Roy model is rejected. In practice, the test for the existence of a non pecuniary component would be carried out by constructing a confidence region according to the methods proposed in Chernozhukov, Lee, and Rosen (2009) or Andrews and Shi (2011) and checking, whether the hyperplane $g(\omega) = 0$ intersects the confidence region. If it does, we fail to reject the hypothesis of existence of a non pecuniary component and if it doesn't, we reject the hypothesis at significance level equal to 1 minus the confidence level chosen for the confidence region. The hypotheses $g \geq 0$ or $g \leq 0$ may be tested in the same way.

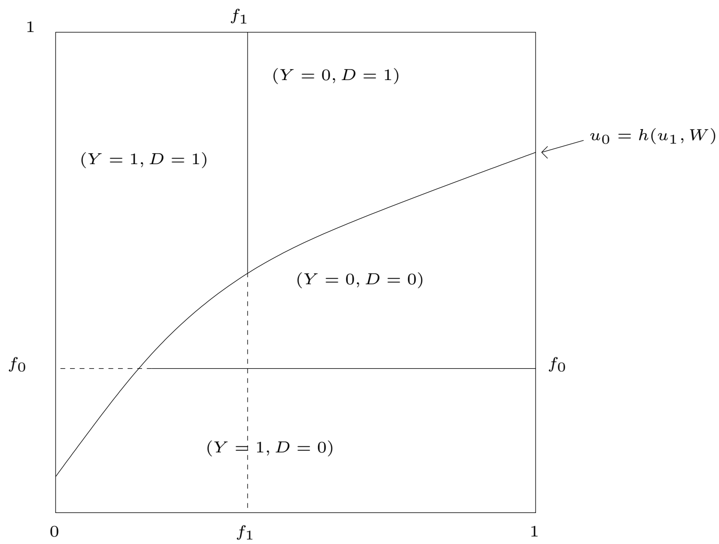
2.2.5. *Sharp bounds without weak separability of the selection function.* The same arguments can be applied to derive the empirical content of the model where the selection equation generalizes Assumption 5 with the following.

Assumption 6 (Nonseparable selection function)**.** *Suppose the selection rule is* $D = 1\{u_0 > h(u_1, W)\}$ *and* h *strictly increasing in* u_1*, for all* W*.*

Assumption 5 is a special case of Assumption 6, where $h(u_1, W) = u_1 + f_0(X_0) - f_1(X_1) + g(W)$. The identified region for the pair $(f_1(x_1), f_0(x_0))$ is obtained in the same way as the weakly separable case except that f_1 attains $\mathbb{P}(1,1) + \mathbb{P}(1,0) + \mathbb{P}(0,0)$ and f_0 attains $\mathbb{P}(1,1) + \mathbb{P}(1,0) + \mathbb{P}(0,0)$. This occurs because the nonlinearity of the curve separating region $D = 1$ from region $D = 0$ allows all the mass corresponding to $\mathbb{P}(0,0)$ to be shifted on the left of f_1, as in Figure 5.

Proposition 4 (Sharp bounds for the extended Roy model without weak separability)**.** *Under Assumptions 1-3 and 6, the identified set for* (f_1, f_0) *is characterized by the following inequalities,*

FIGURE 5. Characterization of the empirical content of the binary Roy model in the unit square of the (u_1, u_0) space without weak separability of the selection function.



where λ *takes the values* 1 *or* 0*:*

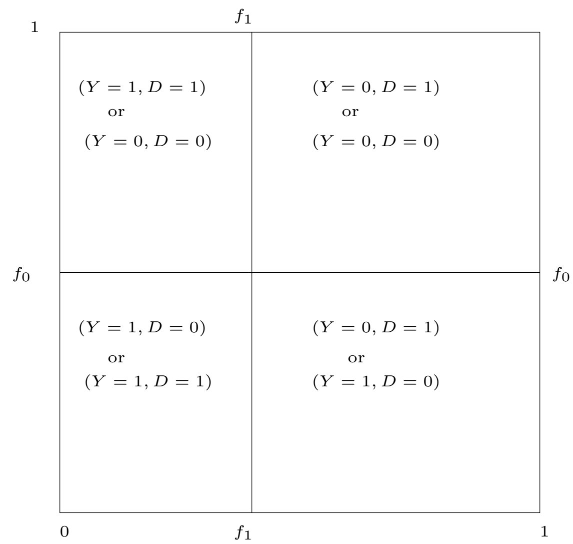
$$\begin{aligned} \mathbb{P}(1,1|\omega) &\leq f_1(x_1) \leq \mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) + \lambda \mathbb{P}(0,0|\omega) \\ \mathbb{P}(1,0|\omega) &\leq f_0(x_0) \leq \mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega) + (1-\lambda)\mathbb{P}(0,1|\omega) \end{aligned} \tag{2.7}$$

In this context, however, the Roy selection assumption $D = 1\{Y_1^* > Y_0^*\}$ may not be tested with the strategy developed above.

3. Sharp bounds for the generalized binary Roy model

So far, we have assumed that selection occurs on the basis of success probability and other observable variables. We now turn to the general case, where unobservable heterogeneity, beyond $u_0 - u_1$, may play a role in sectorial selection. Knowledge of (u_1, u_0) now no longer uniquely determines the outcome $(Y = i, D = j)$ as seen on Figure 6. Multiplicity of equilibria and lack of

FIGURE 6. Characterization of the empirical content of the generalized binary Roy model in the unit square of the (u_1, u_0) space.



coherence of the model can be dealt with, however, with the optimal transportation approach of Galichon and Henry (2011) (or equivalently with the random set approach of Beresteanu, Molchanov, and Molinari (2011) as in Chesher, Rosen, and Smolinski (2011)), as shown in the proof of Theorem 1 below.

Theorem 1 (Sharp bounds for the generalized Roy model)**.** *Under Assumption 1-3, the empirical content of the model is characterized by inequalities (3.1)-(3.3) below (writing* $\omega = (z, x_1, x_0)$ *as before).*

$$\sup_{z,x_0} \mathbb{P}(1,1|\omega) \quad \le f_1(x_1) \le \quad 1 - \sup_{z,x_0} \mathbb{P}(0,1|\omega), \tag{3.1}$$

$$\sup_{z,x_1} \mathbb{P}(1,0|\omega) \quad \le f_0(x_0) \le \quad 1 - \sup_{z,x_1} \mathbb{P}(0,0|\omega) \tag{3.2}$$

and

$$\begin{aligned} & \sup_z \max\Big(0, f_0(x_0) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(0,0|\omega)\Big) \\ \leq \;& \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \qquad (3.3) \\ \leq \;& \inf_z \min\Big(\mathbb{P}(Y=1|\omega), f_1(x_1) + f_0(x_0) - \mathbb{P}(Y=1|\omega)\Big). \end{aligned}$$

Theorem 1 is not an operational characterization of the empirical content of the model since the sharp bounds involve the unknown quantity $\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0)$, which, by the normalization of Assumption 2, is exactly the copula of (u_1, u_0). In the case of total ignorance about the copula of (u_1, u_0), after plugging Fréchet bounds $\max(f_1(x_1) + f_0(x_0) - 1, 0) \leq \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \leq \min(f_1(x_1), f_0(x_0))$, inequalities (3.3) are shown to be redundant. Hence we have the following.

Corollary 4. *The identified set for (f_1, f_0) under Assumption 1-3 is characterized by inequalities (3.1) and (3.2).*

In order to sharpen those bounds, we may consider restrictions on the copula for (u_1, u_0) or restrictions on the selection equation. We consider both strategies in turn.

3.1. **Restrictions on selection.** Consider the following selection model, where selection depends on $Y_1^* - Y_0^*$ and $g(Z, X_1, X_0)$ and selection specific unobserved heterogeneity v, which is weakly separable and which is independent of (resp. dependent on) sector specific unobserved heterogeneity (u_1, u_0) under Assumption 7 (resp. Assumption 8). As before, write $W = (Z, X_1, X_0)$.

Assumption 7. *$D = 1\{Y_1^* - Y_0^* > g(W) + v\}$, with $v \perp\!\!\!\perp (u_1, u_0, W)$ and $\mathbb{E}v = 0$ (without loss of generality).*

With $v \perp\!\!\!\perp (u_1, u_0)$, we have $\mathbb{P}(u_d \leq g(z, x_1, x_0) + v + f_1(x_1) - f_0(x_0)|z, x_1, x_0) = \mathbb{E}_v\mathbb{E}[1\{u_d \leq g(z, x_1, x_0) + v - f_1(x_1) + f_0(x_0)\}|z, x_1, x_0, v] = \max(0, g(z, x_1, x_0) - f_1(x_1) + f_0(x_0))$ and it is shown in Corollary 5 that the bounds on $g(\,)$ derived in Section 2 remain valid.

Corollary 5. *Under assumptions 1-3 and 7, (2.5) holds.*

As for the bounds on (f_1, f_0), (2.4) remain valid under specific domain restrictions for v.

Assumption 8. $D = 1\{Y_1^* - Y_0^* > g(W) + v\}$*, with* $v \perp\!\!\!\perp W$*,* $\mathbb{E}v = 0$ *(without loss of generality).*

Note that Assumption 8 is equivalent to assuming the selection equation $D = 1\{h(W) > \eta\}$ with η arbitrarily dependant on (u_1, u_0). Indeed, one can take $h(W) = f_1(X_1) - f_0(X_0) - g(W)$ and $\eta = v + u_1 - u_0$.

Corollary 6. *Under Assumption 1-3 and 8, (3.1) and (3.2) are sharp bounds for the pair* (f_1, f_2)*.*

From Corollary 6, we conclude that the weak separability of the selection specific unobserved heterogeneity term has no empirical content, in the sense that the identified set for (f_1, f_0) is identical to the case, where there is no information on selection. This is related to the lack of empirical content of LATE in Kitagawa (2009) and it is in sharp contrast with the case of no sector specific heterogeneity in Shaikh and Vytlacil (2011), Jun, Pinkse, and Xu (2010) and Mourifié (2011), where the ordering between f_1 and f_0 can be used as identifying information. Indeed, if $f_1 \leq f_0$, we have $f_1 = \mathbb{P}(Y = 1, D = 1) + \mathbb{P}(u_1 \leq f_1, D = 0) \leq \mathbb{P}(Y = 1, D = 1) + \mathbb{P}(u_1 \leq f_0, D = 0)$. The last term is equal to $\mathbb{P}(Y = 1, D = 0)$ if $u_1 = u_0$ but is not identified in the case with sector specific unobserved heterogeneity.

3.2. **Restrictions on the joint distribution of sector specific heterogeneity.**

3.2.1. *Parametric restrictions on the copula.* In case the copula for (u_1, u_0) is parameterized with parameter vector θ, sharp bounds are obtained straightforwardly by replacing $\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0)$ with the parametric version $F(f_1(x_1), f_0(x_0); \theta)$ in (3.3).

3.2.2. *Perfect correlation.* In the case of perfect correlation between the two sector specific unobserved heterogeneity variables, $\mathbb{P}(u_1 \leq f_1(x_0), u_0 \leq f_0(x_0)) = \min(f_1(x_1), f_0(x_0))$ so that the sharp bounds of Theorem 1 specialize to (3.1), (3.2), $\min(f_1(x_1), f_0(x_0)) \leq \inf_z \mathbb{P}(Y = 1|z, x_1, x_0)$ and $\sup_z \mathbb{P}(Y = 1|z, x_1, x_0) \leq \max(f_1(x_1), f_0(x_0))$, which are the bounds derived in Chiburis (2010).

3.2.3. *Independence.* In the special case, where the two sector specific errors are independent of each other $u_1 \perp\!\!\!\perp u_0$, sharp bounds can be derived from Theorem 1 and $\mathbb{P}(u_1 \leq f_1(x_0), u_0 \leq f_0(x_0)) = \mathbb{P}(u_1 \leq f_1(x_1))\mathbb{P}(u_0 \leq f_0(x_0)) = f_1(x_1)f_0(x_0)$. The sharp bounds obtained allow formal tests of the hypothesis of independence of the two unobserved heterogeneity components. This would not be achievable based only on Fréchet bounds (as noted by Tsiatis (1975) in the case of competing risks), as we always have $f_0 + f_1 - 1 \leq f_0 f_1 \leq \min(f_1, f_0)$ when $0 \leq f_1, f_0 \leq 1$.

3.2.4. *Factor structure.* Theorem 1 also allows us to characterize the empirical content of the factor model for sector specific unobserved heterogeneity proposed in Aakvik, Heckman, and Vytlacil (2005).

Assumption 9 (Factor model)**.** *Sector specific unobserved heterogeneity has factor structure* $u_d = \alpha_d u + \eta_d$, $d = 1, 0$, *with* $\mathbb{E}u = 0$, $\mathbb{E}u^2 = 1$ *(without loss of generality) and* $\eta_1 \perp\!\!\!\perp \eta_0 | u$. η_d *is uniformly distributed on* $[0, 1]$ *for* $d = 1, 0$, *conditionally on* u.

This factor specification for sector specific unobserved heterogeneity is particularly appealing in applications to the effects of employment programs. Success in securing a job depends on common unobservable heterogeneity in talent and motivation and sector specific noise. Under Assumptions 1, 3 and 9, we still have $\mathbb{E}[Y_d|z,x_1,x_0] = f_d(x_d)$ and

$$\begin{aligned}\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1,x_0) &= \mathbb{E}_u\mathbb{P}(\eta_1 \leq f_1(x_1) - \alpha_1 u, \eta_0 \leq f_0(x_0) - \alpha_0 u|x_1,x_0,u) \\ &= \mathbb{E}_u\mathbb{P}(\eta_1 \leq f_1(x_1) - \alpha_1 u|x_1,u)\mathbb{P}(\eta_0 \leq f_0(x_0) - \alpha_0 u|x_1,x_0,u) \\ &= f_1(x_1)f_0(x_0) + \alpha_1\alpha_0.\end{aligned}$$

Hence we can obtain sharp bounds on parameters f_1, f_0, α_1 and α_0 as follows.

Corollary 7 (Sharp bounds for the factor model)**.** *Under Assumptions 1, 3 and 9, the empirical content of the model is characterized by (3.1), (3.2) and (writing $\omega = (z,x_1,x_0)$ as before)*

$$\begin{aligned}&\sup_z \max\Big(0, f_0(x_0) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(0,0|\omega)\Big) \\ \leq\;& f_1(x_1)f_0(x_0) + \alpha_1\alpha_0 \\ \leq\;& \inf_z \min\Big(\mathbb{P}(Y=1|\omega), f_1(x_1) + f_0(x_0) - \mathbb{P}(Y=1|\omega)\Big)\end{aligned}$$

We recover the case of independent sector specific heterogeneity variables, when $\alpha_1 = \alpha_0 = 0$.

Conclusion

We have derived sharp bounds in the simple, extended and generalized binary Roy models, including a factor specification proposed by Aakvik, Heckman, and Vytlacil (2005). The bounds are simple enough to lend themselves to existing inference methods for intersection bounds as in Chernozhukov, Lee, and Rosen (2009) and Andrews and Shi (2011). The methods introduced here can be applied to the derivation of nonparametric sharp bounds for the Tobit version of the Roy model as well as in other binary models with several unobserved heterogeneity dimensions, such as entry and participation games.

Appendix A. Proofs

In all the proofs, we use the notation $\omega = (z, x_1, x_0)$. When there is no ambiguity, we shall write $f_1 = f_1(x_1)$, $f_0 = f_0(x_0)$ and $g = g(\omega)$.

A.1. **Proof of Proposition 1.**

A.1.1. *Validity of the bounds.* See main text.

A.1.2. *Sharpness of the bounds.* To show the sharpness of the joint bounds for $f_1(x_1)$ and $f_0(x_0)$, it is sufficient to construct joint distributions for the unobserved heterogeneity vector (u_0^*, u_1^*) such that each of the extreme points of the convex identified region are attained and which is compatible with the observed data in the following sense:

(1) $P(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 1, D = 0|\omega)$,

(2) $P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 1, D = 1|\omega)$,

(3) $P(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 0, D = 0|\omega)$,

(4) $P(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 0, D = 1|\omega)$.

The identified region is a rectangle and its extreme points are $(f_1(x_1) = P(Y = 1, D = 1|\omega), f_0(x_0) = P(Y = 1, D = 0|\omega))$, $(f_1(x_1) = P(Y = 1|\omega), f_0(x_0) = P(Y = 1|\omega))$, $(f_1(x_1) = P(Y = 1|\omega), f_0(x_0) = P(Y = 1, D = 0|\omega))$ and $(f_1(x_1) = P(Y = 1, D = 1|\omega), f_0(x_0) = P(Y = 1|\omega))$. We construct a joint distribution such that the first three are attained. The last extreme point can be treated identically to the third.

Consider the following candidate density function $f(u_0^*, u_1^*)$ with values:

$$\frac{1}{1-\mathbb{P}(Y=1,D=1|\omega)} \text{ if } u_1^* \geq P(Y=1,D=1|\omega), u_0^* \leq P(Y=1,D=0|\omega),$$

$$0 \text{ if } u_1^* \leq P(Y=1,D=1|\omega), u_0^* \leq P(Y=1,D=0|\omega),$$

$$\frac{1}{1-P(Y=1,D=0|\omega)} \text{ if } u_1^* \leq P(Y=1,D=1|\omega), u_0^* \geq P(Y=1,D=0|\omega),$$

$$\frac{2P(Y=0,D=1|\omega)}{(1-\mathbb{P}(Y=1,D=0|\omega))^2} \text{ if } u_1^* \geq P(Y=1,D=1|\omega),$$

$$u_1^* \leq u_0^* + P(Y=1,D=1|\omega) - P(Y=1,D=0|\omega),$$

$$\frac{2P(Y=0,D=0|\omega)}{(1+P(Y=1,D=0|\omega)-2P(Y=1,D=1|\omega))(1-P(Y=1,D=0|\omega))}$$

$$\text{if } u_0^* \geq P(Y=1,D=0|\omega), u_1^* \geq u_0^* + P(Y=1,D=1|\omega) - P(Y=1,D=0|\omega).$$

It is easy to verify that this function is the density of a joint distribution which is compatible with the observed data (i.e., it respects Conditions 1 to 4) when $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y=1,D=1|\omega)$ and $f_0(x_0) = P(u_0^* \leq f_0(x_0)|x_0) = P(Y=1,D=0|\omega)$.

Now we propose another joint distribution compatible with the observed data such that: $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y=1|\omega)$ and $f_0(x_0) = P(u_0^* \leq f_0(x_0)|x_0) = P(Y=1|\omega)$.

Consider the candidate density function $f(u_0^*, u_1^*)$ with values:

$$
\begin{aligned}
&0 \;\text{ if }\; (u_1^* \geq \mathbb{P}(Y=1|\omega) \;\text{ and }\; u_0^* \leq P(Y=1|\omega)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or }\; (u_1^* \leq \mathbb{P}(Y=1|\omega) \;\text{ and }\; u_0^* \geq P(Y=1|\omega)), \\
&\frac{2P(Y=0,D=0|\omega)}{(1-\mathbb{P}(Y=1|\omega))^2} \;\text{ if }\; P(Y=1|\omega) \leq u_0^* \leq u_1^*, \\
&\frac{2P(Y=0,D=1|\omega)}{(1-\mathbb{P}(Y=1|\omega))^2} \;\text{ if }\; P(Y=1|\omega) \leq u_1^* \leq u_0^*, \\
&\frac{P(Y=1,D=1|\omega)}{\mathbb{P}(Y=1|\omega)^2} \;\text{ if }\; u_1^* \leq u_0^* \leq \mathbb{P}(Y=1|\omega), \\
&\frac{P(Y=1,D=0|\omega)}{\mathbb{P}(Y=1|\omega)^2} \;\text{ if }\; u_0^* \leq u_1^* \leq \mathbb{P}(Y=1|\omega).
\end{aligned}
$$

Again, this function is the density of a joint distribution which is compatible with the observed data (i.e., respects Conditions 1 to 4) when $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1|\omega)$ and $f_0(x_0) = P(u_0^* \leq f_0(x_0)|x_0) = P(Y = 1|\omega)$.

Finally, we propose another joint distribution compatible with the observed data such that: $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega)$ and $f_0(x_0) = P(u_0^* \leq f_0(x_0)|x_0) =$

$P(Y=1|\omega)$. Consider the candidate density function $f(u_0^*, u_1^*)$ with values:

$$
\begin{aligned}
&0 \text{ if } u_1^* \leq P(Y=1, D=1|\omega) \text{ and } (u_0^* \geq P(Y=1|\omega) \\
&\qquad\qquad\qquad \text{or } u_1^* \geq u_0^* + P(Y=1, D=1|\omega) - P(Y=1|\omega)), \\
&\frac{P(Y=1, D=0|\omega)}{(1-P(Y=1, D=1|\omega))P(Y=1|\omega)} \text{ if } u_1^* \geq P(Y=1, D=1|\omega), u_0^* \leq P(Y=1|\omega), \\
&\frac{2P(Y=0, D=1|\omega)}{(1-\mathbb{P}(Y=1|\omega))^2} \text{ if } P(Y=1, D=1|\omega) \leq u_1^* \leq u_0^* + P(Y=1, D=1|\omega) - P(Y=1|\omega), \\
&\frac{2P(Y=0, D=0|\omega)}{(1+P(Y=1|\omega) - 2P(Y=1, D=1|\omega))(1-P(Y=1|\omega))} \\
&\qquad\qquad \text{if } P(Y=1|\omega) \leq u_0^* \leq u_1^* - P(Y=1, D=1|\omega) + P(Y=1|\omega) \\
&\frac{2P(Y=1, D=1|\omega)}{(1-P(Y=1, D=1|\omega))^2} \text{ if } u_1^* + P(Y=1|\omega) - P(Y=1, D=1|\omega) \leq u_0^* \leq P(Y=1|\omega).
\end{aligned}
$$

Again, this function is the density of a joint distribution which is compatible with the observed data (i.e., respects Conditions 1 to 4) when $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega)$ and $f_0(x_0) = P(u_0^* \leq f_0(x_0)|x_0) = P(Y = 1|\omega)$.

This completes the proof.

A.2. **Proof of Proposition 2.** To show validity of the bounds, we drop all the conditioning variables $\omega = (z, x_1, x_0)$ from the notation. We have $D = 1 \Rightarrow Y_0^* + g \leq Y_1^* \Rightarrow 1\{Y_0^* + g \geq 0\} \leq 1\{Y_1^* \geq 0\} \Rightarrow 1\{Y_0^* + g \geq 0\}1\{D = 1\} \leq 1\{Y_1^* \geq 0\}1\{D = 1\} \Rightarrow \mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 1] \leq \mathbb{E}[1\{Y_1^* \geq 0\}|D = 1] \Rightarrow \mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 1] \leq \mathbb{E}[Y_1|D = 1]$. We can easily derive equivalent inequalities when $D = 0$. Hence, if $D = 1\{Y_1^* > Y_0^* + g\}$ then $\mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 1] \leq \mathbb{E}[Y_1|D = 1]$ and $\mathbb{E}[Y_1|D = 0] \leq \mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 0]$. Hence, when $g \geq 0$, $\mathbb{E}[Y_0|D = 1] \leq \mathbb{E}[Y_1|D = 1]$ and when $g \leq 0$, $\mathbb{E}[Y_1|D = 0] \leq \mathbb{E}[Y_0|D = 0]$. Finally, if $g = 0$ we have $\mathbb{E}[Y_d|D = d] \geq \mathbb{E}[Y_d|D = 1-d]$ where $d \in \{0, 1\}$. Those inequalities allow us to construct the sharp bounds for f_1 and f_0 in the case where $D = 1\{Y_1^* > Y_0^* + g\}$. Indeed, $f_1 = \mathbb{E}[Y_1] = \mathbb{E}[Y_1, D = 1] + \mathbb{E}[Y_1|D = 0]P(D = 0)$ and $f_0 = \mathbb{E}[Y_0] = \mathbb{E}[Y_0, D = 0] + \mathbb{E}[Y_0|D = 1]P(D = 1)$. Now, if $g \geq 0$, then $P(Y = 1, D = 1) \leq f_1 \leq P(Y = 1, D = 1) + P(D = 0)$ and $P(Y = 1, D = 0) \leq f_0 \leq P(Y = 1)$. On the other hand, if $g \leq 0$, $P(Y = 1, D = 1) \leq f_1 \leq P(Y = 1)$ and $P(Y = 1, D = 0) \leq f_0 \leq P(Y = 1, D = 0) + P(D = 1)$. Finally, $f_0 = \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \geq u_0 + f_1 - f_0 - g\}] + \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \leq u_0 + f_1 - f_0 - g\}]$. Hence, if $g \geq f_1$, then $\{u_1 \leq u_0 + f_1 - f_0 - g\} \Rightarrow \{u_0 \geq f_0\}$ and $f_0(X_0) \leq \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \geq u_0 + f_1 - f_0 - g\}] + \mathbb{E}[1\{u_0 \leq f_0\}1\{u_0 \geq f_0\}] \leq \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \geq u_0 + f_1 - f_0 - g\}] = P(Y = 1, D = 0)$.

Now the bounds for g can be obtained as follows.

- If $g + f_0 - f_1 \geq 0$ and $g \leq f_1$, then $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq u_1 + g + f_0 - f_1\}$ and $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq f_0\}$. So $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq u_1 + g + f_0 - f_1\} \cap \{u_0 \leq f_0\}$. Hence $g + f_0 - f_1 = P(u_0 \leq g + f_0 - f_1) \leq P(\{u_0 \leq u_1 + g + f_0 - f_1\} \cap \{u_0 \leq f_0\}) = P(Y = 1, D = 0)$.
- If $g + f_0 - f_1 \geq 0$ and $g \geq f_1$, then $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq u_1 + g + f_0 - f_1\}$, hence $g + f_0 - f_1 = P(u_0 \leq g + f_0 - f_1) \leq P(\{u_0 \leq u_1 + g + f_0 - f_1\}) = P(D = 0)$. As $f_0 = P(Y = 1, D = 0)$ we have $g - f_1 \leq P(Y = 0, D = 0)$.
- If $g + f_0 - f_1 \leq 0$ and $g \geq -f_0$, then by similar arguments, we have $g + f_0 - f_1 \geq -P(Y = 1, D = 1)$.
- If $g + f_0 - f_1 \leq 0$ and $g \leq -f_0$, then $g + f_0 \geq -P(Y = 0, D = 1)$.

Finally, the validity of bounds $f_1(x_1) - \mathbb{P}(1, 1|\omega) - \mathbb{P}(1, 0|\omega) - \mathbb{P}(0, 1|\omega) \leq g(\omega) \leq \mathbb{P}(1, 1|\omega) + \mathbb{P}(1, 0|\omega) + \mathbb{P}(0, 0|\omega) - f_0(x_0)$ is shown formally in the main text. This completes the proof.

A.3. **Proof of Proposition 3.** As previously, our method consists in constructing joint distributions for (u_1, u_0) such that all points of the identified set for (f_1, f_0) are attained. All points in the identified set of Proposition 1 can be attained as shown in the proof of Proposition 1.

There remains to show that all points in the rectangle with corners $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega)+\min(0,\mathbb{P}(0,0|\omega)-\varepsilon),\mathbb{P}(1,0|\omega))$, $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega)+\min(0,\mathbb{P}(0,0|\omega)-\varepsilon),\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega))$, $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega),\mathbb{P}(1,0|\omega))$ and $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega),\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega))$ for $\varepsilon > 0$ arbitrarily small (and symmetrically all points in the rectangle with corners $(\mathbb{P}(1,1|\omega),\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega)+\min(0,\mathbb{P}(0,1|\omega)-\varepsilon))$, $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega),\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega)+\min(0,\mathbb{P}(0,1|\omega)-\varepsilon))$, $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega),\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega))$ and $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega),\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega)))$.

Compatibility between the joint distribution and the observed data can be expressed as follows:

(1) $P(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y=1, D=0|\omega)$,

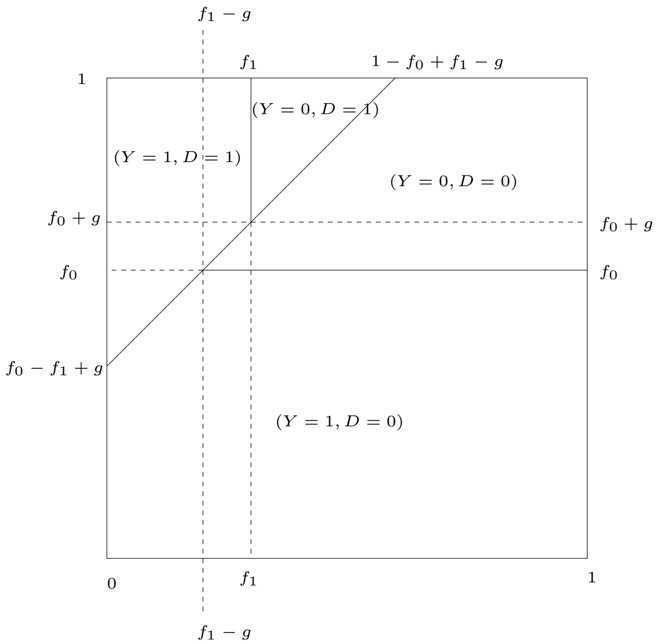
(2) $P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y=1, D=1|\omega)$,

(3) $P(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y=0, D=0|\omega)$,

(4) $P(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y=0, D=1|\omega)$,

The method of proof is illustrated in Figure 7. Assume that $\mathbb{P}(0,0|\omega) > 0$ (otherwise the rectangle treated below collapses). We construct a joint distribution for (u_1, u_0) such that $f_1(x_1) = (\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega)+\mathbb{P}(0,0|\omega)-\alpha_1$ and $f_0(x_0) = \mathbb{P}(1,0|\omega)+\alpha_0$, for any (α_1, α_0) satisfying $0 < \alpha_1 \leq \mathbb{P}(0,0|\omega)$ and $0 \leq \alpha_0 \leq \mathbb{P}(1,1\omega)$.

FIGURE 7. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $f_0(x_0) > f_1(x_1)$, $0 < g(\omega) < f_1(x_1)$ and $f_0(x_0) + g(\omega) < 1$.



Consider the function $f(u_0^*, u_1^*)$ with values:

$$
\begin{array}{lll}
0 & \text{if} & f_0(x_0) \leq u_0^* \leq u_1^* + g(\omega) + f_0(x_0) - f_1(x_1) \\
 & & \quad \text{and } 1 - f_0(x_0) + f_1(x_1) - g(\omega) \geq u_1^* \geq f_1(x_1), \\
\frac{2(\mathbb{P}(0,0|\omega) - \alpha_1)}{g(\omega)^2} & \text{if} & f_0(x_0) \leq u_0^* \leq u_1^* + g(\omega) + f_0(x_0) - f_1(x_1), u_1^* \leq f_1(x_1), \\
\frac{2\alpha_2}{(f_1(x_1) - g(\omega))^2} & \text{if} & f_0(x_0) \geq u_0^* \geq u_1^* + g(\omega) + f_0(x_0) - f_1(x_1), \\
\frac{2(\mathbb{P}(1,1|\omega) - \alpha_2)}{2f_1(x_1)(1 - f_0(x_0)) - g(\omega)^2} & \text{if} & u_0^* \geq f_0(x_0), u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2\mathbb{P}(0,1|\omega)}{(1 - f_0(x_0) - g(\omega))} & \text{if} & f_1(x_1) \leq u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2\mathbb{P}(1,0|\omega)}{2f_1(x_1)f_0(x_0) - (f_1(x_1) - g(\omega))^2} & \text{if} & f_1(x_1) \geq u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
0 & \text{if} & u_1^* \geq f_1(x_1), u_0^* \leq f_0(x_0), \\
\frac{\alpha_1}{(1 - f_0(x_0))(f_0(x_0) - f_1(x_1) + g(\omega))} & \text{if} & f_0(x_0) \leq u_0^*, u_1^* \geq 1 - f_0(x_0) + f_1(x_1) - g(\omega),
\end{array}
$$

It is easy to verify that this function is a density of a joint distribution which is compatible with the observed data (i.e respects conditions 1 to 4) and such that $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega) + P(D = 0|\omega)$ and $g(\omega) + f_0(x_0) - f_1(x_1) = P(u_0^* \leq g(\omega)+f_0(x_0)-f_1(x_1)|\omega) = P(Y = 1, D = 0|\omega)$, when $g(\omega)$ is set equal to $\alpha_1 - f_0(x_0)+f_1(x_1)$.

Symmetrically, we can show that any point in the rectangle with corners $(\mathbb{P}(1,1|\omega), \mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega)+\min(0, \mathbb{P}(0,1|\omega)-\varepsilon))$, $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega), \mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega)+\min(0, \mathbb{P}(0,1|\omega)-\varepsilon))$, $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega), \mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega))$ and $(\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega), \mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega))$ can be attained and this completes the Proof.

A.4. **Proof of Proposition 4.** The proof is exactly identical to the that of Proposition 3 except that $h(u_1, \omega)$ can be chosen as in Figure 5 so that all the mass $\mathbb{P}(0,0|\omega)$ can be shifted on the left of $f_1(x_1)$ and therefore we can no longer restrict α_1 to be strictly positive. The case $\alpha_1 = 0$ is also attained. The result follows immediately.

A.5. **Proof of Theorem 1.** Under Assumptions 1-3, the model can be equivalently written $(Y, D) \in G((u_1, u_0)|W)$ almost surely conditionally on $W = (Z, X_1, X_0)$, where G is a multi-valued mapping, which to (u_1, u_0) associates $(y, d) = G((u_1, u_0)|W) = \{(1,1),(1,0)\}$ if $u_1 \leq f_1(x_1)$ and $u_0 \leq f_0(x_0)$, $\{(0,1),(1,0)\}$ if $u_1 > f_1(x_1)$ and $u_0 \leq f_0(x_0)$, $\{(1,1),(0,0)\}$ if $u_1 \leq f_1(x_1)$ and $u_0 > f_0(x_0)$ and $\{(0,1),(0,0)\}$ if $u_1 > f_1(x_1)$ and $u_0 > f_0(x_0)$. Hence Theorem 1 of Galichon and Henry (2011) applies and the empirical content of the model is characterized by the collection of inequalities $P(A|W) \leq P((u_1, u_0) : \; G((u_1, u_0)|W) \text{ hits } A|W)$ for each subset A of $\{(0,0),(0,1),(1,0),(1,1)\}$ (i.e., 16 inequalities). The only non redundant inequalities are $P(1,1|W) \leq f_1(X_1)$, $P(1,0|W) \leq f_0(X_0)$, $P(0,1|W) \leq 1 - f_1(X_1)$, $P(0,0|W) \leq 1 - f_0(X_0)$, $P(Y = 0|W) \leq 1 - P(u_1 \leq f_1(X_1), u_0 \leq f_0(X_0)|X_1, X_0)$,

$P(Y=1|W) \leq 1-P(u_1 > f_1(X_1), u_0 > f_0(X_0)|X_1,X_0)$, $P(0,0|W)+P(1,1|W) \leq P(u_1 \leq f_1(X_1), u_0 \leq f_0(X_0)|X_1,X_0) + P(u_0 > f_0(X_0)|X_0)$ and $P(0,1|W)+P(1,0|W) \leq P(u_1 \leq f_1(X_1), u_0 \leq f_0(X_0)|X_1,X_0) + P(u_1 > f_1(X_1)|X_1)$. After some manipulation, the result follows.

A.6. **Proof of Corollary 5.** We show that the bounds (2.5) for g remain valid. We drop conditioning variables from the notation throughout this section.

- If $g+v+f_0-f_1 \geq 0$ and $g+v \leq f_1$, then $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq u_1+g+v+f_0-f_1\}$ and $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq f_0\}$. So $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq u_1+g+v+f_0-f_1\} \cap \{u_0 \leq f_0\}$. Therefore $P(u_0-v \leq g+f_0-f_1) \leq P(\{u_0 \leq u_1+g+v+f_0-f_1\} \cap \{u_0 \leq f_0\}) = P(Y=1,D=0)$.
- If $g+v+f_0-f_1 \geq 0$ and $g+v \geq f_1$, then $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq u_1+g+v+f_0-f_1\}$. Therefore $P(u_0-v \leq g+f_0-f_1) \leq P(\{u_0 \leq u_1+g+v+f_0-f_1\}) = P(D=0)$.
- If $g+v+f_0-f_1 \leq 0$ and $g+v \geq -f_0$, then $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq u_0+f_1-f_0-g-v\}$ and $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq f_1\}$. So $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq u_0+f_1-f_0-g-v\} \cap \{u_1 \leq f_1\}$. Therefore $P(u_1+v \leq f_1-f_0-g) \leq P(\{u_1 \leq u_0+f_1-f_0-g-v\} \cap \{u_1 \leq f_1\}) = P(Y=1,D=1)$.
- If $g+v+f_0-f_1 \leq 0$ and $g+v \leq -f_0$, then $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq u_0+f_1-f_0-g-v\}$. Hence $P(u_1+v \leq f_1-f_0-g) \leq P(u_1 \leq u_0+f_1-f_0-g-v) = P(D=1)$.

Now, since $v \perp\!\!\!\perp (u_0,u_1)$, we have: $P(u_0 \leq g+v+f_0-f_1) = \mathbb{E}_v[\mathbb{E}[1\{u_0 \leq g+v+f_0-f_1\}|v]] = \mathbb{E}_v[g+v+f_0-f_1] = g+f_0-f_1$. Then, we get the following:

- If $g+v+f_0-f_1 \geq 0$ and $g+v \leq f_1$, then $g+f_0-f_1 \leq P(Y=1,D=0)$.

- If $g + v + f_0 - f_1 \geq 0$ and $g + v \geq f_1$, then $g - f_1 \leq P(Y = 0, D = 0)$.
- If $g + v + f_0 - f_1 \leq 0$ and $g + v \geq -f_0$, then $g + f_0 - f_1 \geq -P(Y = 1, D = 1)$.
- If $g + v + f_0 - f_1 \leq 0$ and $g + v \leq -f_0$, then $g + f_0 \geq -P(Y = 0, D = 1)$.

which completes the proof.

A.7. **Proof of Corollary 6.** Our goal here is to show that the following bounds are sharp for f_0 and f_1.

$$
\begin{aligned}
P(Y = 1, D = 1|\omega) &\leq f_1(x_1) \leq P(Y = 1, D = 1|\omega) + P(D = 0|\omega) \\
P(Y = 1, D = 0 \mid \omega) &\leq f_0(x_0) \leq P(Y = 1, D = 0|\omega) + P(D = 1|\omega).
\end{aligned}
$$

The previous results show that the lower bounds are sharp. Now, to show that these bounds are sharp for $f_1(x_1)$ it is sufficient to construct a joint distribution (u_0^*, u_1^*) such that $f_1(x_1)$ equals $P(Y = 1, D = 1|\omega) + P(D = 0|\omega)$ and $f(0, x) = P(Y = 1, D = 0|\omega) + P(D = 1|\omega)$ and which is compatible with the observed data in the following sense:

(1) $P(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 1, D = 0|\omega)$
(2) $P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 1, D = 1|\omega)$
(3) $P(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 0, D = 0|\omega)$
(4) $P(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 0, D = 1|\omega)$

Define the following joint distribution (u_0^*, u_1^*, v^*) such that $u_0^* + u_1^* \leq f_1(x_1) + f_0(x_0)$ and $2v^* = 3u_0^* - 3u_1^* - 3f_0(x_0) + 3f_1(x_1) - 2g(\omega)$. Under the condition that $u_0^* + u_1^* \leq f_1(x_1) + f_0(x_0)$, we have $\{u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v^*\} \Rightarrow \{u_1^* \leq f_1(x_1)\}$ and

$\{u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v^*\} \Rightarrow \{u_0^* \leq f_0(x_0)\}$. Hence,

$$\begin{aligned} f_1(x_1) &= P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &\qquad +P(u_1^* \leq f_1(x_1), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &= P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &\qquad +P(u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &= P(Y = 1, D = 1|\omega) + P(D = 0|\omega). \end{aligned}$$

With the same strategy, we can also show $f_0(x_0) = P(Y = 1, D = 0|\omega) + P(D = 1|\omega)$.

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